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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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OFFICE ... No. 23 FEB ... MAD...

The Inverse of the Miquel-Clifford Configuration

By E. H. Neville.

In the absence of any reference in Coolidge's exhaustive *Treatise on the Circle and the Sphere*, it is to be assumed that the effect of inverting the configuration associated with the names of Miquel and Clifford has escaped notice.

The classical theorem may be enunciated as follows:

With any even number of lines in a plane is associated a point, called their Miquel point, and with any odd number is associated a circle, called their Clifford circle. If the number of lines in a set is an odd number m, the omission of these lines one at a time from the set gives m subsets each of which is even, and the Miquel points of these subsets are concyclic on the Clifford circle of the original set. If the number of lines in a set is an even number n, the omission of these lines one at a time gives n subsets each of which is odd, and the Clifford circles of these subsets are concurrent in the Miquel point of the original set.

The Miquel point of two lines is their point of intersection The Clifford circle of three lines passes through the point of intersection of any two of them, that is, is the circumcircle of the triangle which they form. The theorem that *the circumcircles of the four triangles formed by four lines are concurrent* was enunciated by Steiner in 1827 and may well have been known much earlier, since it is so obvious a corollary to the theorem that *the circumcircle of a triangle formed by three tangents to a parabola passes through the focus*, that it can hardly have escaped notice for 66 years after the publication of the latter theorem by Lambert; the point of concurrence of the circumcircles is the Miquel point of the four lines. In 1836 Miquel proved that the points of concurrence associated with the five quadrilaterals derived from five lines are concyclic, and finally Clifford in 1870 demonstrated that the alternation of concurrent circles and concyclic points persists indefinitely. Meanwhile Miquel had shewn that the theorem attributed to Steiner and

his own theorem concerning five lines remain true if four concurrent circles are substituted for the four lines in the one or five concurrent circles for the five lines in the other. The extension is of course obvious from the theory of inversion, but this theory was not known then to Miquel, and his proofs deal directly with circles. Clifford's proof, which is reproduced in substance in Salmon's *Higher Plane Curves*, depends on focal properties of curves of the nth class, and it is essential to this argument that the figure should be based on lines, not on circles; Clifford wrote without consulting Miquel's own papers, and apparenty without knowing Miquel's later enunciations, and perhaps it is to the fact that he did not think of Miquel's result in terms of circles at all that we owe the brilliant proof which led him to his generalisation.

It we invert the Miquel-Clifford configuration of any order from an arbitrary point, we have a figure consisting wholly of circles, and the remarkable property of the figure is that it is homogeneous, in a sense that is easy to understand: although circles enter the construction at stages further and further removed from the concurrent circles which replace the straight lines of Clifford's figure, ultimately they are all on the same footing. Moreover the distinction between the case of an odd number and the case of an even number disappears, and we can enunciate the result corresponding to Clifford's in the form that

For any value of r, there is a homogeneous configuration of 2^r circles and 2^r points such that $r + 1$ of the circles pass through each of the points and $r + 1$ of the points lie on each of the circles.

It is naturally to be recognised that the 2^r points do not include every point of intersection of any two of the circles: they are the points of multiple intersection by which the figure is characterised. The figure can be completed from the $r + 1$ circles through any one of the essential points, but is not determined by the $r + 1$ points which lie on one circle.

Proof by enumeration, Clifford's result being assumed, is trivial and need not detain us. But to devise a notation for the points and circles is to discover an independent proof that the configuration is possible, and is to reach Clifford's result by a route more direct than his,

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Let us denote a point by a row of symbols each of which is either α or β, and a circle by a similar row in which each symbol is either A or B, and let a point lie on a circle if their symbols agree or if they differ in only one place. Thus for a figure with two circles A, B, the points of intersection are α, β and more generally if two circles are denoted by XAY, XBY, where X and Y are rows of A's and B's, the points of intersection of these two circles are denoted by $x\alpha y$, $x\beta y$, where x is derived from X and y from Y by the substitution of α for A and β for B throughout. If the symbols of two circles differ in two places, the points of intersection can still be denoted: XAYAZ cuts XBYBZ in $x\alpha y\beta z$ and $x\beta y\alpha z$, and XAYBZ cuts XBYAZ in $x\alpha y\alpha z$ and $x\beta y\beta z$. But if the symbols of the circles differ in more places than two, the points of intersection are not among the points which the notation can identify.

If we denote two circles through a point $\alpha\alpha$ by AA and BA, we can denote a third circle through the point by AB. Then the second intersection of the first two circles must be denoted by $\beta\alpha$, the second intersection of AA and AB by $\alpha\beta$ and the second intersection of BA and AB by $\beta\beta$. Consistently with the property of the notation we can denote the circle through the three points $\beta\alpha$, $\alpha\beta$, $\beta\beta$ by BB, and when this circle is introduced we have a 4-circle figure in which there are 3 circles through each of the 4 marked points and 3 points marked on each of the 4-circles The circle through three of the points may be called the Clifford circle of the fourth point; the symbols of a point and of its Clifford circle disagree in every place.

We next extend the symbols in a 4-circle figure by denoting the points by $\alpha\alpha\alpha$, $\beta\alpha\alpha$, $\alpha\beta\alpha$, $\beta\beta\alpha$ and the circles by AAA, BAA, ABA, BBA. We can then introduce another circle through $\alpha\alpha\alpha$, denoting it by AAB. Each set of 3 circles through $\alpha\alpha\alpha$ forms a triangle, and in each case we have appropriate symbols for the vertices of this triangle and for its circumcircle. Thus the triangle formed by AAA, BAA, AAB has vertices $\beta\alpha\beta$, $\alpha\alpha\beta$, $\beta\alpha\alpha$ and circumcircle BAB, and the triangle formed by BAA, ABA, AAB has vertices $\alpha\beta\beta$, $\beta\alpha\beta$, $\beta\beta\alpha$ and circumcircle BBB. The four 4-circle figures determined by the four triangles

require 8-circles, and in their construction from $\alpha\alpha\alpha$ they involve 7 points, including $\alpha\alpha\alpha$ itself. The circles exhaust the possible 3-letter symbols, the points use all the symbols except $\beta\beta\beta$, and in the construction incidence of the 7 points upon the 8 circles is expressed as the notation directs. Up to this point we have assumed only that two circles which cut once cut again and that three points determine a circle; now we require the theorem enunciated by Miquel, that the circumcircles of the four triangles formed by the 4 circles through $\alpha\alpha\alpha$ are concurrent. The four circumcircles are being denoted by ABB, BAB, BBA, BBB, and in assigning to the point of concurrence the one unappropriated 3-letter symbol $\beta\beta\beta$. we maintain the characteristic feature of the notation.

We have now an 8-circle figure with its relations of concurrence exhibited completely. It is important to notice that the figure can be reconstructed from the four circles through any one of its 8 points, and that the symbols for the points and circles of the whole figure are determined by the symbols of the four concurrent circles which are used. Each point in the figure is in four 4-circle figures, and its Clifford circles in these figures are concurrent in a point which can be called its Miquel associate in the 8-circle figure; the symbols of a point and of its Miquel associate differ in every place.

Again we extend first the notation and then the figure, adding an α to the symbol of each point and an A to the symbol of each circle in an 8-circle figure and introducing another circle AAAB through the point whose symbol has become $\alpha\alpha\alpha\alpha$. The five circles through $\alpha\alpha\alpha\alpha$ provide five sets of four circles, and each of these determines an 8-circle figure. Two sets of four circles through $\alpha\alpha\alpha\alpha$ have in common one set of three circles through the same point, and the 4-circle figure determined by this set is common to the 8-circle figures determined by the two sets of four. Thus the five 8-circle figures involve in addition to the five circles through $\alpha\alpha\alpha\alpha$ only ten other circles, which are the Clifford circles of $\alpha\alpha\alpha\alpha$ in the ten 4-circle figures determined by the ten sets of three which can be chosen from the five given circles; the symbol for one of these ten Clifford circles is fixed by the set of three

which determines the 4-circles figure, and as so fixed is valid in each of the 8-circle figures of which that 4-circle figure forms a part: for example, the Clifford circle of αααα for ABAA, AABA, AAAB is ABBB, and this symbol is valid both in the 8-circle figure determined by these three circles and AAAA and in the 8-circle figure determined by these three circles and BAAA. The 15-circles which compose the five 8-circle figures exhibit 16 points of intersection, namely, the point αααα, the 10 second intersections of the circles through αααα, and the Miquel associates of αααα in the 5 figures; the 16 points utilise all the possible 4-letter symbols, the 15 ci·cles utilise all except BBBB, and the relations of incidence are in agreement with the notation.

It remains to shew that the five Miquel points, whose symbols are αβββ, βαββ, ββαβ, βββα, ββββ, are in fact concyclic. For this purpose we have only to construct a figure from the five circles through the second intersection of two of the circles through αααα. The five circles determine five 8-circle figures combined in a figure with 15-circles and 16 points; since the 4-letter symbols are adequate to describe the points, and are all needed for this purpose, the points are the same as in the figure constructed from circles through αααα; of the 15 symbols needed for the circles, 14 must be already in use and must refer to the same circles as before, and since the symbol not needed differs in every place from the symbol for the point used as initial point, and the point is not αααα, the svmbol is not BBBB. In other words, in the new construction there is a circle BBBB on which the 5 Miquel points of αααα in the original construction are known to lie, and with the insertion of this circle we have the 16-circle figure.

The argument has been left in general terms in order that the continuation to higher orders may be evident, but the fact that no additional theorem of concurrence is involved is perhaps clearer if we follow the argument in detail with a particular initial point. Of the circles through βααα, BBAA, BABA, BAAB form a triangle whose vertices are βαββ, ββαβ, βββα, and the circumcircle of this triangle passes both through the Miquel associate of βααα in the 8-circle figure determined by adding AAAA to the group and through the Miquel

associate of $\beta\alpha\alpha\alpha$ in the 8-circle figure determined by adding BAAA to the group; the first of these Miquel associates is $\alpha\beta\beta\beta$, the second is $\beta\beta\beta\beta$, and the use of BBBB as a symbol for the circumcircle is consistent with the symbols for the 5 points through which the circle is now known to pass.

It is superfluous to set out formally the proof by induction of the general theorem. When the existence of a 2^{r-1}-circle figure is known, we form the $r+1$ figures of this kind determined by the groups of r circles selected from $r+1$ circles through a point P. Any point or circle which is common to two of these 2^{r-1}-circle figures belongs to a 2^{r-2}-circle figure which they have in common, and therefore has a definite symbol which is consistent with the part played by the point or circle in both of the 2^{r-1}-circle figures. The $r+1$ subsidiary figures utilise all but one of the available symbols; the unwanted symbol is a point-symbol if r is odd, a circle-symbol if r is even. If r is odd, P has a Clifford circle in each of the subsidiary figures and we require to know that these $r+1$ circles are concurrent: if r is even, P has a Miquel associate in each of the subsidiary figures and we require to know that these $r+1$ points are concyclic. In each case we infer the result by rebuilding the figure from the second intersection of two of the circles through P. If this new origin is Q, the symbol which was not previously wanted can not differ in *every* place from that of Q, since it differs in every place from that of P; it follows that this symbol is now involved, that is to say, that the concurrence or concyclicality which remains to be established when the figure is constructed from P must be a synthesis of some of those established at earlier stages when the figure is constructed from Q.

If S is any one of the points of concurrence, a 2^r-circle figure may be described with reference to S as follows. The circles are the $\binom{r+1}{1}$ circles through S, the Clifford circles of S in the $\binom{r+1}{3}$ 4-circle figures determined by 3 of the $r+1$ circles, the Clifford circles of S in the $\binom{r+1}{5}$ 16-circle figures determined by 5 of the $r+1$ circles and so on. The points are S itself, the second intersections of the $\binom{r+1}{2}$ pairs that can be selected from the $r+1$ circles, the Miquel associates of S

in the $\binom{r+1}{4}$ 8-circle figures determined by 4 of the $r+1$ circles, the Miquel associates of S in the $\binom{r+1}{6}$ 32-circle figures determined by 6 of the $r+1$ circles, and so on.

The proof given here of Clifford's theorem illustrates well the possibilities latent in notation. The proof of the theorem that a 16-circle figure can be built on five concurrent circles is essentially the same as Miquel's, and nothing but the increasing complexities of the figure can have prevented Miquel from anticipating Clifford. But valuable though it is here, the notation has a very narrow range of usefulness, and it is easy to describe figures in which it must not or cannot be introduced. If A, B, C, D are four points on a circle, and if any four circles are drawn through A and B, through B and C, through C and D, and through D and A, the four second intersections A′, B′, C′, D′ of the two circles through A, the two through B, the two through C, and the two through D, are concyclic. Here we have 6 circles and 8 points; there are 3 circles through each of the points and 4 of the points on each circle, and it is easy to see that the circles all have the same relation to the complete figure But if we attempt to denote the points and circles by 3 letter symbols, the notation suggests that the two groups of points AB′CD′ and A′BC′D should be concyclic, though usually they are not. Moreover it is possible so to choose the circles determining the figure that the groups of points AA′CC′ and BB′DD′ are both concyclic; when the figure has this form it is an 8 circle figure in which 4 circles pass through each of 8 points and 4 of these points lie on each of the circles, but the structure of the figure in which two circles play a different part from the other six, is entirely different from that of the Miquel 8-circle figure, and the notation of this paper is not misleading but literally inapplicable.

The Direct Evaluation of Poisson's Brackets

By S. Narasimhachari.

The equations for the variation of the elliptic elements $a, e, i, \theta, \varpi, \epsilon,$ in the Planetary Theory, have been obtained in various ways. The object of the paper is to point out a much easier method of obtaining them.

If the disturbing function R be expressed in terms of the elliptic elements, and if A_r, A_s be any two of these elements, the equation for the variation of A_r is

$$\frac{dA_r}{dt} = \sum_s \left\{ A_s, A_r \right\} \frac{\partial R}{\partial A_s} \quad \dots \quad \dots \quad (1)$$

where $\{ A_s, A_r \}$ is a Poisson's bracket, the value of which is defined by

$$\left\{ A_s, A_r \right\} = \sum_i \frac{\partial (A_s, A_r)}{\partial (\alpha_i, \beta_i)}$$

the α's and β's being canonical variables.*

In order to obtain the value of $\{ A_s, A_r \}$ Plummer adopts the following procedure:—He first established (§ 130) the relation between Lagrange's and Poisson's Brackets, namely, that the determinants formed by the two sets of brackets are reciprocal. He then calculates Lagrange's Brackets $[A_s, A_r]$ where A_s, A_r are any pair of the six elements $a, e, i, \theta, \varpi, \epsilon,$ (§ 137). From the relationship between the two sets of Brackets the values of $\{ A_s, A_r \}$ are deduced in § 138 and it only remains to substitute these values in (1) to obtain the equations for the variation of elements.

The essential feature of the method adopted in this paper, is that the Poisson's Brackets are calculated directly, without bringing in any knowledge of Lagrange's Brackets.

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* H. C. Plummer: *Dynamical Astronomy*, § 139.

The canonical variables α_i, β_i are given by the equations

$$\left.\begin{aligned} \alpha &= \varepsilon - \varpi, & \beta &= \sqrt{\mu a}, \\ \alpha_2 &= \omega - \theta, & \beta_2 &= \sqrt{\mu a\,(1-e^2)}, \\ \alpha_3 &= \theta, & \beta_3 &= \sqrt{\mu a\,(1-e^2)}\cos i \end{aligned}\right\} \quad \ldots (2)$$

Solving the equations (2) for the elliptic elements we have at once

$$\left.\begin{aligned} a &= \frac{\beta^2}{\mu}, \quad e = \sqrt{1-\frac{\beta_2^{\,2}}{\beta^2}}, \quad i = \cos^{-1}\frac{\beta_3}{\beta_2} \\ \theta &= \alpha_3, \quad \varpi = \alpha_2 + \alpha_3, \quad \varepsilon = \alpha + \alpha_2 + \alpha_3 \end{aligned}\right\} \quad \ldots (3)$$

Now by definition

$$\left\{A_s, A_r\right\} = \frac{\partial(A_s, A_r)}{\partial(\alpha, \beta)} + \frac{\partial(A_s, A_r)}{\partial(\alpha_2, \beta_2)} + \frac{\partial(A_s, A_r)}{\partial(\alpha_3, \beta_3)}$$

Since a is a function of β only

$$\left\{a, A_r\right\} = -\frac{2\beta}{\mu}\,\frac{\partial A_r}{\partial \alpha}$$

(A) $\left\{ \begin{array}{l} \therefore\ \{a, A_r\} = 0, \text{ if } A_r \text{ is independent of } \alpha, \text{ i.e., if } A_r \neq \varepsilon \\ \left\{a, \varepsilon\right\} = -\dfrac{2\beta}{\mu} = -2\sqrt{\dfrac{a}{\mu}}. \end{array} \right.$

Since e is a function of β_2 and β only

$$\left\{e, A_r\right\} = -\frac{1}{e}\,\frac{\beta_2^{\,2}}{\beta^3}\cdot\frac{\partial A_r}{\partial \alpha} + \frac{1}{e}\cdot\frac{\beta_2}{\beta^2}\,\frac{\partial A_r}{\partial \alpha_2}$$

(B) $\left\{ \begin{array}{l} \{e, A_r\} = 0, \text{ if } A_r \neq \varpi \text{ or } \varepsilon \\ \left\{e, \varpi\right\} = \dfrac{\sqrt{1-e^2}}{e\sqrt{\mu a}} \\ \left\{e, \varepsilon\right\} = \dfrac{\sqrt{1-e^2}}{e\sqrt{\mu a}}\left(1-\sqrt{1-e^2}\right) \end{array} \right.$

Since i is a function of β_3 and β_2 alone

$$\left\{i, A_r\right\} = -\frac{\beta_3}{\beta_2\sqrt{\beta_2^{\,2}-\beta_3^{\,2}}}\cdot\frac{\partial A_r}{\partial \alpha_2} + \frac{1}{\sqrt{\beta_2^{\,2}-\beta_3^{\,2}}}\,\frac{\partial A_r}{\partial \alpha_3}$$

(C)

$$\therefore \{i, A_r\} = 0, \text{ if } A_r = a, e, \text{ or } i.$$

$$\{i, \theta\} = \frac{1}{\sqrt{\mu a(1-e^2)}} \frac{1}{\sin i}$$

$$\{i, \varpi\} = \{i, \varepsilon\} = \frac{\tan i/2}{\sqrt{\mu a(1-e^2)}}$$

(D)

$$\{\theta, A_r\} = 0. \text{ if } A_r \neq i, \text{ since } \theta \text{ depends on } \alpha_3 \text{ only.}$$

$$\{\theta, i\} = -\{i, \theta\} = -\frac{1}{\sqrt{\mu a(1-e^2)}} \frac{1}{\sin i}$$

(E)

$$\{\varpi, A_r\} = 0, \text{ if } A_r \neq e \text{ or } i, \ \varpi \text{ being a function of } \alpha_2 \text{ and } \alpha_3 \text{ only.}$$

$$\{\varpi, e\} = -\{e, \varpi\} = -\frac{\sqrt{1-e^2}}{e\sqrt{\mu a}}$$

$$\{\varpi, i\} = -\{i, \varpi\} = -\frac{\tan i/2}{\sqrt{\mu a(1-e^2)}}$$

(F)

$$\{\varepsilon, A_r\} = 0, \text{ if } A_r \neq a, e, \text{ or } i, \text{ because } \varepsilon \text{ depends only on } \alpha, \alpha_2 \text{ and } \alpha_3.$$

$$\{\varepsilon, a\} = -\{a, \varepsilon\} = 2\sqrt{\frac{a}{\mu}}.$$

$$\{\varepsilon, e\} = -\{e, \varepsilon\} = -\frac{\sqrt{1-e^2}}{e\sqrt{\mu a}}\left(1-\sqrt{1-e^2}\right)$$

$$\{\varepsilon, i\} = -\{i, \varepsilon\} = -\frac{\tan i/2}{\sqrt{\mu a(1-e^2)}}.$$

The substitution in (1) of the values of the Poisson's Brackets given by equations (A), gives the equation for the variation of a, those in (B) give $\frac{de}{dt}$, those in (C) give $\frac{di}{dt}$ and so on, $\frac{d\varepsilon}{dt}$ being given by the set (F). We thus obtain the following equations for the variation of the elliptic elements:—

$$\frac{da}{dt} = 2\sqrt{\frac{a}{\mu}} \cdot \frac{\partial R}{\partial \varepsilon}.$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{e\sqrt{\mu a}} \left\{ \frac{\partial R}{\partial \varpi} + \left(1 - \sqrt{1-e^2}\right) \frac{\partial R}{\partial \varepsilon} \right\}$$

$$\frac{di}{dt} = -\frac{1}{\sqrt{\mu a\,(1-e^2)}} \left[\frac{1}{\sin i} \frac{\partial R}{\partial \theta} + \tan\frac{i}{2} \left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \varepsilon} \right) \right]$$

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{\mu a\,(1-e^2)}} \frac{1}{\sin i} \frac{\partial R}{\partial i}.$$

$$\frac{d\varpi}{dt} = \frac{1}{\sqrt{\mu a}} \left[\frac{\sqrt{1-e^2}}{e} \frac{\partial R}{\partial e} + \frac{\tan i/2}{\sqrt{1-e^2}} \cdot \frac{\partial R}{\partial i} \right]$$

$$\frac{d\varepsilon}{dt} = -2\sqrt{\frac{a}{\mu}} \frac{\partial R}{\partial a} + \frac{1}{\sqrt{\mu a}} \left[\frac{\sqrt{1-e^2}}{e} \left(1 - \sqrt{1-e^2}\right) \frac{\partial R}{\partial e} + \frac{\tan i/2}{\sqrt{1-e^2}} \cdot \frac{\partial R}{\partial i} \right].$$

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The Evaluation of Two Functional Determinants with applications.*

BY M. BHIMASENA RAO AND M. VENKATARAMA AYYAR.

1. Consider the determinant

$$D \equiv \begin{vmatrix} f_{1,1}-f_{1,2n} & f_{1,2}-f_{1,2n-1} & \cdots & f_{1,n}-f_{1,n+1} \\ f_{2,1}-f_{2,2n} & f_{2,2}-f_{2,2n-1} & \cdots & f_{2,n}-f_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n,1}-f_{n,2n} & f_{n,2}-f_{n,2n-1} & & f_{n,n}-f_{n,n+1} \end{vmatrix}$$

where $f_{r,s}$ is a function of the integral variable s of the $2r$th or $(2r-1)$th degree, so that

$$f_{r,s+1}-f_{r,s}=\Delta f_{r,s}$$

$$f_{r,s+2}-2f_{r,s+1}+f_{r,s}=\Delta^2 f_{r,s}$$

and so on. Multiply D by the determinant

$$\begin{vmatrix} 0 & 0 & . & . & 0 & 1 \\ 0 & 0 & . & 0 & 1 & -3 \\ 0 & 0 & . & 1 & -5 & 10 \\ . & . & . & . & . & . \\ 1 & -{}_{2n-1}C_1 & . & . & & (-1)^{n-1}\,{}_{2n-1}C_{n-1} \end{vmatrix}$$

The value of this multiplier is easily seen to be $(-1)^{\frac{n(n-1)}{2}}$.

Performing the multiplication row by row, we get

$$(-)^{\frac{n(n-1)}{2}}.D=\begin{vmatrix} -\Delta f_{1,n} & -\Delta f_{2,n} & \cdots & -\Delta f_{n,n} \\ -\Delta^3 f_{1,n-1} & -\Delta^3 f_{2,n-1} & \cdots & -\Delta^3 f_{n,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ -\Delta^{2n-1} f_{1,1} & -\Delta^{2n-1} f_{2,1} & \cdots & -\Delta^{2n-1} f_{n,1} \end{vmatrix}$$

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* Read at the 13th, Annual Meeting of the Indian Science Congress, Bombay, 1925.

i.e.,

$$(-1)^{\frac{n(n-1)}{2}}.D=(-1)^n \begin{vmatrix} \Delta f_{1,n} & \Delta f_{2,n} & \cdots & \Delta f_{n,n} \\ \Delta^3 f_{1,n-1} & \Delta^3 f_{2,n-1} & \cdots & \Delta^3 f_{n,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^{2n-1} f_{1,1} & \Delta^{2n-1} f_{2,1} & \cdots & \Delta^{2n-1} f_{n,1} \end{vmatrix}$$

But, by hypothesis, $f_{r,s}$ is a function of s of the $2r$th or $(2r-1)$th degree.

$\therefore \quad \Delta f_{r,s}$ is of degree $(2r-1)$ or $(2r-2)$.

$\Delta^2 f_{r,s}$ is of degree $(2r-2)$ or $(2r-3)$

. . . .

$\Delta^{2r} f_{r,s}$ is a constant or zero,

and $\quad \Delta^{2r+1} f_{r,s} = \Delta^{2r+2} f_{r,n} = \ldots \ldots = 0.$

Hence $\quad \Delta^3 f_{1,n} = \Delta^4 f_{1,n} = \ldots \ldots = 0$

$\Delta^5 f_{2,n} = \Delta^6 f_{2,n} = \ldots \ldots = 0$

and so on.

$$\therefore (-1)^{\frac{n(n-1)}{2}} D=(-1)^n \begin{vmatrix} \Delta f_{1,n} & \Delta f_{2,n} & . & . & \Delta f_{n,n} \\ 0 & \Delta^3 f_{2,n-1} & . & . & \Delta^3 f_{n,n-1} \\ 0 & 0 & \Delta^5 f_{3,n-2} & . & \Delta^5 f_{n,n-2} \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & 0 & \Delta^{2n-1} f_{n,1} \end{vmatrix}$$

$$\therefore \quad D = (-1)^{\frac{n(n+1)}{2}} . \Delta f_{1,n} \, \Delta^3 f_{2,n-1} \, \Delta^5 f_{3,n-2} \ldots \Delta^{2n-1} f_{n,1} \ldots$$

2. Now, consider the determinant

$$D' \equiv \begin{vmatrix} f_{1,1} + f_{1,2n-1} & f_{1,2} + f_{1,2n-2} & . & f_{1,n} \\ f_{2,1} + f_{2,2n-1} & f_{2,2} + f_{2,2n-2} & . & f_{2,n} \\ . & . & . & . \\ . & . & . & . \\ f_{n,1} + f_{n,2n-1} & f_{n,2} + f_{n,2n-2} & . & f_{n,n} \end{vmatrix}$$

where, now, $f_{r,s}$ stands for a polynomial in the integral variable s of degree $(2r-1)$ or $(2r-2)$; so that

$$\Delta^{2r} f_{r,s} = \Delta^{2r+1} f_{r,s} = \dots \dots = 0.$$

Multiply D′ by

$$\begin{vmatrix} 0 & 0 & . & 0 & 0 & 1 \\ 0 & 0 & . & 0 & 1 & -2 \\ 0 & 0 & . & 1 & -4 & +6 \\ . & . & . & . & . & . \\ 1 & -{}_{2n-2}C_1 & . & . & & (-1)^{n-1}\,{}_{2n-2}C_{n-1} \end{vmatrix}$$

The value of the multiplier is easily seen to be $(-1)^{\frac{n(n-1)}{2}}$.

Performing the multiplication as before, row by row,

$$(-1)^{\frac{n(n-1)}{2}}.D' = \begin{vmatrix} f_{1,n} & f_{2,n} & . & . & f_{n,n} \\ \Delta^2 f_{1,n-1} & \Delta^2 f_{2,n-1} & . & . & \Delta^2 f_{n,n-1} \\ . & . & . & . & . \\ \Delta^{2n-2} f_{1,1} & \Delta^{2n-3} f_{2,1} & . & . & \Delta^{2n-2} f_{n,1} \end{vmatrix}$$

$$= \begin{vmatrix} f_{1,n} & f_{2,n} & . & . & f_{n,n} \\ 0 & \Delta^2 f_{2,n-1} & . & . & \Delta^2 f_{n,n-1} \\ 0 & 0 & \Delta^4 f_{3,n-2} & . & \Delta^4 f_{n,n-2} \\ . & . & . & . & . \\ 0 & 0 & . & . & \Delta^{2n-2} f_{n,1} \end{vmatrix}$$

$$\therefore D' = (-1)^{\frac{n(n-1)}{2}} . f_{1,n} . \Delta^2 f_{2,n-1} . \Delta^4 f_{3,n-2} \dots \Delta^{2n-2} f_{n,1} \quad \dots \text{(II)}$$

3. With the help of these two results, the values of a number of persymmetric determinants whose elements are the co-efficients in Bessel's Functions (multiplied by some constants) and whose values have been reduced to determinants involving co-efficients in binomial expansions are very easily found. It may be noted here that though a number of such determinants involving binomial co-efficients have been investigated by Zeppel and others, the above types have not been dealt

with. Such Persymmetrics have been worked out by us in a long paper awaiting publication. Therein the notation for a persymmetric is that of enclosing within rectangular brackets the first and the last elements of the first row (or, what is the same thing, the first column), *e.g.*,

$$[{}_{2n+1}A_{2n},\ {}_{2n+1}A_{2n+m-1}] = \begin{vmatrix} {}_{2n+1}A_{2n} & {}_{2n+1}A_{2n+1} & \cdot & {}_{2n+1}A_{2n+m-1} \\ {}_{2n+1}A_{2n+1} & {}_{2n+1}A_{2n+2} & \cdot & {}_{2n+1}A_{2n+m} \\ \cdot & \cdot & \cdot & \cdot \\ {}_{2n+1}A_{2n+m-1} & \cdot & \cdot & {}_{2n+1}A_{2n+2m-2} \end{vmatrix}$$

where the values of the 'A's in the case of $a_n = 1$ which makes the M-function $M_n(x) = \frac{n+1}{x} \cdot J_{n+1}(2x)$ have been given in a table appended to the paper on 'Determinants involving specified numbers' in this *Journal* Vol. 14, page 138

4. In the course of our work, we have shown that

$[{}_{2n+1}A_{2n},\ {}_{2n+1}A_{2n+m-1}]$ reduces when $m \equiv 1 \pmod{2n+1}$ to

$$(-1)^{mn-n+n(n-1)/2} \begin{vmatrix} {}_{m}C_0 + {}_{m+2n-2}C_0 & {}_{m+1}C_0 + {}_{m+2n-3}C_0 & \cdot & {}_{m+n-1}C_0 \\ {}_{m+1}C_2 + {}_{m+2n-1}C_2 & \cdot & \cdot & {}_{m+n}C_2 \\ \cdot & \cdot & \cdot & \cdot \\ {}_{m+n-1}C_{2n-2} + {}_{m+3n-3}C_{2n-2} & \cdot & \cdot & {}_{m+2n-2}C_{2n-2} \end{vmatrix}$$

Here the determinant is of the type D′ in § 2 and satisfies the conditions required therein. Hence its value is

$$(-1)^{\frac{n(n-1)}{2}} \cdot {}_{m+n-1}C_0 \cdot \Delta^2\, {}_{m+n-1}C_2 \cdot \Delta^4\, {}_{m+n-1}C \times \ldots \Delta^{2n-2}\, {}_{m+n-1}C_{2n-2}$$

$$= (-1)^{\frac{n(n-1)}{2}} \cdot ({}_{m+n-1}C_0)^n$$

$$= (-1)^{\frac{n(n-1)}{2}}.$$

$$\therefore\ [{}_{2n+1}A_{2n},\ {}_{2n+1}A_{2n+m-1}] = (-1)^{n(n-1)} \qquad \ldots (a)$$

when $m \equiv 1 \pmod{2n+1}$.

The same persymmetric in the case when $m \equiv n \pmod{2n+1}$ reduces to

$$(-)^{mn+n}.\begin{vmatrix} {}_{m+3}C_2 - {}_{m+2n}C_2 & {}_{m+4}C_2 - {}_{m+2n-1}C_2 & \cdots & {}_{m+n+1}C_2 - {}_{m+n+2}C_2 \\ {}_{m+4}C_4 - {}_{m+2n+1}C_4 & {}_{m+5}C_4 - {}_{m+2n}C_4 & \cdots & {}_{m+n+2}C_4 - {}_{m+n+3}C_4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ {}_{m+n+1}C_{2n-2} - {}_{m+3n-2}C_{2n-2} & \cdot & \cdot & \cdot \end{vmatrix}$$

Here the determinant is of the type D and satisfies the conditions specified therein. The order of the determinant is now $(m-1)$ and its value is

$$(-)^{\frac{(n-1)n}{2}} . \Delta_{m+n+1}C_2 . \Delta^3{}_{m+n+1}C_4 \ \ldots \ \Delta^{2n-3}{}_{m+n+1}C_{2n-2}$$

$$= (-)^{\frac{n(n-1)}{2}} \left({}_{m+n+1}C_1\right)^{n-1}.$$

$$\therefore \ [{}_{2n+1}A_{2n}, {}_{2n+1}A_{2n+m-1}] = (-1)^{m+n+\frac{n(n+1)}{2}} . (m+n+1)^{n-1} \ \ldots \ (b)$$

when $m \equiv n \pmod{2n+1}$.

Again, for the same persymmetric, take the case when $m \equiv n+2 \pmod{2n+1}$. Here the value is

$$(-)^{mn+n-1}\begin{vmatrix} {}_{m+1}C_2 - {}_{m+2n-2}C_2 & . & {}_{m+n-1}C_2 - {}_{m+n}C_2 \\ {}_{m+2}C_4 - {}_{m+2n-1}C_4 & . & {}_{m+n}C_4 - {}_{m+n+1}C_4 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ {}_{m+n-1}C_{2n-2} - {}_{m+3n-4}C_4 & . & . \end{vmatrix}$$

Here again the determinant got is of the form D and hence its value is

$$(-1)^{\frac{n(n-1)}{2}} . \Delta_{m+n-1}C_2 . \Delta^3{}_{m+n-1}C_4 \ \ldots \ \Delta^{2n-3}{}_{m+n-1}C_{2n-2}$$

$$= (-1)^{\frac{n(n-1)}{n}} . ({}_{m+n-1}C_1)^{n-1}.$$

$$\therefore \ [{}_{2n+1}A_{2n}, {}_{2n+1}A_{2n+m-1}] = (-)^{mn+\frac{(n-1)(n-2)}{2}} . (m+n-1)^{n-1} \ldots \ (c)$$

Lastly, in the same persymmetric, when $m = 0$, (mod. $2n + 1$), a determinant of the type D′ is got. Then

$$[{}_{2n+1}A_{2n},\ {}_{2n+1}A_{2n+m-1}]$$

$$= (-1)^{n(m-1)+\frac{n(n+1)}{2}} . (-1)^{\frac{n(n-1)}{2}} . ({}_{m+n}C_0)^n$$

$$= (-1)^{mn} \qquad \ldots \quad (d)$$

5. Now, let us take the values got for $[{}_{2n+1}A_{2n+1},\ {}_{2n+1}A_{2n+m}]$.

In the case when $m = n$ mod ($2n + 1$), we get

$$\begin{vmatrix} {}_{m+2}C_2 - {}_{m+2n+1}C_2 & \cdots & {}_{m+n+1}C_2 - {}_{m+n+2}C_2 \\ {}_{m+3}C_4 - {}_{m+2n+2}C_4 & \cdots & {}_{m+n+2}C_4 - {}_{m+n+3}C_4 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ {}_{m+n}C_{2n} - {}_{m+3n}C_{2n} & \cdots & \cdots \end{vmatrix}$$

which is of the type D.

∴ From (I) its value is

$$(-1)^{\frac{n(n+1)}{2}} . \Delta\ {}_{m+n+1}C_2 . \Delta^3\ {}_{m+n+1}C_4 \ \ldots\ \Delta^{2n-1}\ {}_{m+n+1}C_{2n}.$$

∴ $[{}_{2n+1}A_{2n+1},\ {}_{2n+1}A_{2n+m}]$ when $m = n$ (mod. $2n + 1$)

$$= (-)^{mn+\frac{n(n+1)}{2}} . (m + n + 1)^n. \qquad \ldots \quad (e)$$

Similarly in the cases when $m = n + 1$ (mod. $2n + 1$) we again get a determinant of the type D and when $m = 0$ mod. ($2n + 1$), a determinant of the type D′. Using the results (I) and (II), we get

$$[{}_{2n+1}A_{2n+1},\ {}_{2n+1}A_{2n+m}] = (-1)^{mn+\frac{n(n-1)}{2}} (m + n)_n. \qquad \ldots \quad (f)$$

when $m = n + 1$ (mod. $2n + 1$) while for $m = 0$ mod. ($2n + 1$) its value is $(-1)^{mn}$ ... ... (g)

In the same manner, we may apply our results to simplify persymmetrics of the type

$$[{}_{2n+2}A_{2n+1},\ {}_{2n+2}A_{2n+m}] \text{ and } [{}_{2n+2}A_{2n+2},\ {}_{2n+2}A_{2n+m+1}].$$

Without however multiplying such instances, we shall proceed to verify the results (*a*) to (*e*), thereby showing also the brevity and efficacy of the method employed. The actual 'A' numbers specified above are given in the appendix to the paper on 'Determinants involving specified numbers' in this *Journal*, Vol. XIV, No 4.

6. Let us verify for $n = 2$. Here the elements

$${}_{2n+1}A_{2n},\quad {}_{2n+1}A_{2n+1},\quad {}_{2n+1}A_{2n+2}\ \cdots\ \cdots$$

are 1, 5, 20, 75, 275, 1001, 3640, 13260, 48450, 177650.

For direct calculation, the numbers which occur are very large and the task of evaluation of the determinants correspondingly tedious. We will, therefore, add a method for the simplification of persymmetric determinants involving such big numbers. In Scott's Determinants, it is shown that

$$\begin{vmatrix} u_1 & u_2 & \cdot & \cdot & u_n \\ u_2 & u_3 & \cdot & \cdot & u_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n & u_{n+1} & \cdot & \cdot & u_{2n-1} \end{vmatrix} = \begin{vmatrix} u_1 & \Delta u_1 & \cdot & \cdot & \Delta^{n-1} u_1 \\ \Delta u_1 & \Delta^2 u_1 & \cdot & \cdot & \Delta^n u_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^{n-1} u_1 & \Delta^n u_1 & \cdot & \cdot & \Delta^{2n-2} u_1 \end{vmatrix}$$

where Δ is the usual difference-operator. But, merely taking differences does not simplify much. So we will introduce a constant multiplier before taking the differences. Denoting the first of the above determinants by U we get

$$U = \begin{vmatrix} u_1 & u_2 - ku_1 & u_3 - 2ku_2 + k^2 u_1 & \cdot\cdot \\ u_2 & u_3 - ku_2 & u_4 - 2ku_3 + k^2 u_2 & \cdot\cdot \\ u_3 & u_4 - ku_3 & u_5 - 2ku_4 + k^2 u_3 & \cdot\cdot \\ \cdot & \cdot & \cdot \quad \cdot & \cdot\cdot \\ u_n & u_{n+1} - ku_n & u_{n+2} - 2ku_{n+1} + k^2 u_n & \cdot\cdot \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 - ku_1 & \cdot & \cdot \\ u_2 - ku_1 & u_3 - 2ku_2 + k^2u_1 & \cdot & \cdot \\ u_3 - 2ku_2 + k^2u_1 & u_4 - 3ku_3 + 3k^2u_2 - k^3u_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

so that if we write

$$u_2 - ku_1 = \Delta_{(k)} u_1$$

$$u_3 - 2ku_2 + k^2 u_1 = \Delta^2_{(k)} u_1,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

then

$$U = \begin{vmatrix} u_1 & \Delta_{(k)} u_1 & \cdot & \cdot & \Delta^{n-1}_{(k)} u_1 \\ \Delta_{(k)} u_1 & \Delta^2_{(k)} u_1 & \cdot & \cdot & \Delta^n_k u_1 \\ \cdot & \cdot & \cdot & \cdot & , \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^{n-1}_{(k)} u_1 & \Delta^{n-1}_{(k)} u_1 & \cdot & \cdot & \Delta^{2n-2}_{(k)} u_1 \end{vmatrix} \qquad \ldots \text{(III)}$$

For the series of A-numbers mentioned above, taking $u_1 = 1$ $k = 3$, the differences are

$$2, -1, \ 3, -4, \ 8, -14, \ 27, \ -51, \ 98, \ldots$$

and if we take $u_1 = 5$, $k = 3$, they are

$$5, 0, 5, -4, 10, -15, 30, -55, \ldots$$

7. Taking result (a) and putting $n = 2$, $m = 1$, evidently

$${}_5A_4 = 1 = (-1)^{2(1-1)}.$$

To apply result (b), let $m = 2$.

Then $\quad [{}_5A_4, {}_5A_5] = (-1)^{2.2 + \frac{2.3}{2}} (2 + 2 + 1)^{2-1}$ from (b)

$$= -5.$$

By direct substitution, $[{}_5A_4, {}_5A_5] = \begin{vmatrix} 1 & 5 \\ 5 & 20 \end{vmatrix} = -5$ agreeing with

the above. Again, taking $m = 4$, we have, from (c)

$$[{}_5A_4,\ {}_5A_7] = (-1)^{4.2+\frac{(2-1)(2-2)}{2}}.(4+2-1)^{2-1}$$
$$= 5.$$

Directly,

$$[{}_5A_4,\ {}_5A_7] = \begin{vmatrix} 1 & 5 & 20 & 75 \\ 5 & 20 & 75 & 275 \\ 20 & 75 & 275 & 1001 \\ 75 & 275 & 1001 & 3640 \end{vmatrix}$$ which, on using III,

$$= \begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & -4 \\ -1 & 3 & -4 & 8 \\ 3 & -4 & 8 & -14 \end{vmatrix} = 5$$

agreeing with the above.

Again, let us take $m = 5$, that is $\equiv 0 \pmod{5}$.

Then, from (d),

$$[{}_5A_4,\ {}_5A_8] = (-1)^{5.2} = 1.$$

Directly,

$$[{}_5A_4,\ {}_5A_8] = \begin{vmatrix} 1 & 5 & 20 & 75 & 275 \\ 5 & 20 & 75 & 275 & 1001 \\ 20 & 75 & 275 & 1001 & 3640 \\ 75 & 275 & 1001 & 3640 & 13260 \\ 275 & 1001 & 3640 & 13260 & 48450 \end{vmatrix}$$

which on applying result (III)

$$= \begin{vmatrix} 1 & 2 & -1 & 3 & -4 \\ 2 & -1 & 3 & -4 & 8 \\ -1 & 3 & -4 & 8 & -14 \\ 3 & -4 & 8 & -14 & 27 \\ -4 & 8 & -14 & 27 & -51 \end{vmatrix}$$

This works out to 1 agreeing with the above.

In this manner, results (*e*), (*f*) and (*g*) can also be verified.

We shall content ourselves with only showing the working in the case of (*g*). As before take $n = 2$ so that $2n + 1 = 5$ and take $m = 5$, that is, $m \equiv 0$ (mod. 5).

Now from (*g*)

$$[{}_5A_5,\ {}_5A_9] = (-)^{mn} = (-1)^{5.2} = 1.$$

Directly

$$[{}_5A_5,\ {}_5A_9] = \begin{vmatrix} 5 & 20 & 75 & 2'5 & 1001 \\ 20 & 75 & 275 & 1001 & 3640 \\ 75 & 2.5 & 1001 & 3640 & 13260 \\ 275 & 1001 & 3640 & 13260 & 48450 \\ 1001 & 3640 & 13260 & 48450 & 177650 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 5 & 0 & 5 & -4 \\ 5 & 0 & 5 & -4 & 10 \\ 0 & 5 & -4 & 10 & -15 \\ 5 & -4 & 10 & -15 & 30 \\ -4 & 10 & -15 & 30 & -55 \end{vmatrix}$$

on applying (III), with $k = 3$,

This works out to 1, agreeing with the result got from (*g*).

Some Diophantine Equations*

By Balakram.

1. *Summary*: Denote by $f(k)$ the equation

$$x^2 - ny^2 = k$$

where n is a given positive integer without square factors, k a given positive or negative integer, and x, y integers to be found subject to the restriction that they have no common factor. For a particular value of k, the equation cannot be solved for all n, unless $k = 1$. The discussion in this note relates mostly to possible forms of n, for $k = -1, 2$, and -2. It is known that for no value of n can there be solutions for more than one of these values of k † This is generalized in § 3 below, and it is proved that one, but not more, of a certain group of values of k, furnishes solutions for a given n. The question "which of these k's furnishes solutions?" has not been answered except for special forms of n. The case n a prime or twice a prime, say $2N$, is completely worked out in §§ 5, 6, except when $N = 8p + 1$; but the results cannot be new, as they are almost obvious from the theory of quadratic residues. If $N = 8p + 1$, it can always be written as

$$N = \alpha^2 + 16\beta^2$$

α being odd. It is known ‡, that if $N \equiv 9 \pmod{16}$ with $(2/N)_4 = -1$, *i.e.* if $\alpha \equiv \pm 3 \pmod{8}$ with β odd, $f(-1)$ has solutions, and that if $N \equiv 1 \pmod{16}$ with $(2/N)_4 = +1$, *i.e.* $\alpha \equiv \pm 1 \pmod{8}$ with β even, $f(-1)$ may have solutions. This result is generalised in § 10, and the cases of all N's, except $N = (8p \pm 1)^2 + 64\gamma^2$ solved completely. For this form of N, the particular value of k, $k = -1$, 2, or -2, is ascertained by actual calculation up to $N = 3761$ (§ 11).

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* Presented to the Third Conference of the Indian Mathematical Society and since revised and enlarged. See this *Journal*, Vol. XIII, p. 86.

† *Vide* Dickson's *History of the Theory of Numbers*, Vol. II, p. 369.

‡ *Vide* Dickson, Vol. II, pp. 370, 386.

2. To express $\sqrt{n}$, as a continued fraction, we obtain certain integers a_r and k_r as follows :—

$$\left.\begin{aligned} (\sqrt{n}+a_1)(\sqrt{n}-a_1) &= 1.k_1 \\ (\sqrt{n}+a_2)(\sqrt{n}-a_2) &= k_1.k_2 \\ \ldots \quad \ldots \quad & \ldots \quad \ldots \\ (\sqrt{n}+a_r)(\sqrt{n}-a_r) &= k_{r-1}.k_r \\ \ldots \quad \ldots \quad & \ldots \quad \ldots \\ (\sqrt{n}+a_m)(\sqrt{n}-a_m) &= k_{m-1}.k_m = k_{m-1}.k \end{aligned}\right\} \quad \ldots \quad (2\,1)$$

where the a's and k's are positive, $a_1 = [\sqrt{n}]$ the integral part of $\sqrt{n}$, and the other a's are given by

$$\sqrt{n} - a_{r+1} = \sqrt{n} + a_r - k_r\,[(\sqrt{n}+a_r)/k_r] \quad \ldots \quad (2.2)$$

It is proved in text-books that one of the k_m, say k_m, is unity. We stop the process at k_m. The other k's are each greater than 1 and less than $2\sqrt{n}$, and after a certain stage they repeat themselves in reverse order, so that

$$\left.\begin{aligned} k_{m-1} = k_2;\ \ k_{m-2} = k_3;\ \ldots\ \ldots\ k_{m-r} = k_r, \\ a_m = a_1;\ \ a_{m-1} = a_2;\ \ldots\ \ldots\ a_{m-r} = a_{r+1}. \end{aligned}\right\}$$

involving

If j is any integer $< n$, $f(\pm j)$ has solutions only if j occurs among the k's. If $f(-1)$ has solution, both $f(+j)$ and $f(-j)$ have solutions, otherwise only one of them.

The relation between the a's and the k's ensures that the conjugate continued products

$$(\sqrt{n}+a_1)(\sqrt{n}+a_2) \quad \ldots (\sqrt{n}+a_r)$$

and

$$(-\sqrt{n}+a_1)(-\sqrt{n}+a_2)\ldots(-\sqrt{n}+a_r)$$

should each be divisible by $k_1\,k_2\,\ldots\,k_{r-1}$. Multiplying the two sets and dividing out by $(k_1\,k_2\,\ldots\,k_r)^2$, we get

$$A_r^2 - n\,B_r^2 = (-1)^r\,k_r$$

where A_r and B_r are integers prime to each other.

Putting $r = m$,

$$A_m^2 - n B_m^2 = (-1)^m$$

and there is no other value of r which will make the right-hand side ± 1. Thus if m is odd, $2\sigma + 1$, $f(-1)$ has solutions, and the identities (2.1) are arranged symmetrically about

$$(\sqrt{n} + a_{\sigma+1})(\sqrt{n} - a_{\sigma+1}) = k_\sigma . k_{\sigma+1} = k_\sigma^2.$$

If m is even, say 2ρ, $f(-1)$ has no solutions, and the identities (2.1) are arranged symmetrically about the two middle identities

$$(\sqrt{n} + a_\rho)(\sqrt{n} - a_\rho) = k_{\rho-1} . k_\rho$$

$$(\sqrt{n} + a_\rho)(\sqrt{n} - a_\rho) = k_\rho . k_{\rho-1},$$

as $$a_{\rho+1} = a_\rho \text{ and } k_{\rho+1} = k_{\rho-1}.$$

3. THEOREM: *If n has q odd divisors $< \sqrt{n}$, excluding unity, then solutions exist for one, and only one, of the $4q + 3$ equations*

$$f(-1),\ f(2),\ f(-2),\ f(\nu),\ f(-\nu),\ f(2\nu),\ f(-2\nu)$$

where ν is any one of the divisors.

Denoting the $2q + 1$ numbers $2, \nu, 2\nu$ generally by t, the steps in the proof of the theorem are:—

(i) If m, as defined in §2, is odd, $f(-1)$ has solutions, and conversely;

(ii) If $m = 2\rho$, k_ρ is one of the t's, and $f(k_\rho)$ or $f(-k_\rho)$ has solutions, but not both.

(iii) Conversely, if any of the k's, say k_ρ, is a t, k_ρ is the turning point of the k's, and $m = 2\rho$;

(iv) There can be only one k which is a t.

Of these, (i) is well-known; and (iv) is deduced from (iii) coupled with the fact that m, as defined in § 2, has only one value.

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NOTES AND QUESTIONS.

Notes and Questions.

A Variation of a Familiar Card Trick.

1. There are several tricks prevalent in which the principle involved is the possibility of expressing a number uniquely in a given scale of notation. To those mentioned in books on mathematical recreations may be added, the following, which appears to be well-known in South India.

A magician A goes out of the room, while his confederate B is pointed out a particular square in a chess-board. When A enters the room B communicates with him by three snaps of the fingers, each made by either the right or the left-hand, with the fingers pointing either up or down. Thereupon A is able to point out the square in question.

Since each snap can be made in four ways, and since the number of squares is equal to 64, that is, to $4 \times 4 \times 4$, one sees theoretically the feasibility of the trick. The principle involved is not quite the 'scale of notation principle,' since zero digits are repugnant to the present case. If, for the scale of notation n, we disallow zero digits and admit instead n itself as a digit, then it is clear that numbers from 1 to n^r could be expressed in the form $[p_0 p_1 \cdots p_r]$, where each p is one of the numbers $1, 2, 3 \ldots \ldots n$, *i e.*

$$[p_0 p_1 \cdots p_r] = \{ (p_0 - 1) n^r + (p_1 - 1) n^{r-1} + \ldots + (p_r - 1) \} + 1.$$

The trick can accordingly be performed by taking $n = 4$, and expressing the numbers from 1 to 64 in this shape, and indicating the successive digits by snaps. In practice, this is done by dividing the board into quadrants, and sub-quadrants associated in a definite manner with the numbers 1, 2, 3, 4.

2. A well-known card trick is to deal 27 cards face upward in rows of 3 each—each new row half covering the row already dealt. The spectator is requested to think of a card and to name the column in which it is. The rows are then taken up without disturbing the order of the cards, placed upon one another, and dealt out in rows as before beginning with the left-hand card of each row. The column in which the card thought of, is now situated, being again pointed out, it is clear that it could be identified by a further repetition of the same process.

We are concerned here with a variation of this trick, in which the freedom of arranging the three columns arbitrarily after each deal is restricted by the requirement that the card thought of shall appear in a

specifiic position—say, the twelfth—when the cards are taken up after the final deal. What the performer does in this case is to find $p_0\, p_1\, p_2$, where

$$12 = [p_0 p_1 p_2] = \{ (p_0 - 1).3^2 + (p_1 - 1).3 + (p_2 - 1) \} + 1.$$

Thus, $p_2 = 3$, $p_1 = 1$, $p_0 = 2$.

Then the performer so arranges the columns for the second deal, that the column pointed out in the first deal, comes to occupy the p_2th or third position. Similarly the column pointed out in the second deal, occupies the p_1th or first position for the third deal. Then if the columns are so taken up that the column pointed out in the third deal occupies the p_0th or second position, the card thought of will appear as the $12th$.

For this rule to work, the order of cards in each column should not be disturbed, and when the columns have been placed on one another, the pack should be turned so as to be face downward, and the cards dealt from the top, for the next deal.

3. The theory may be stated with reference to a pack of n^3 dealt in the same way in rows of n each. Let each column be divided into n equal sections from the top; it is then clear that the cards of the column which is p_2th for the second deal will occupy the p_2th sections of the n columns. Hence the cards which occupy p_2th column before the second deal and the p_1th before the third deal, will be the p_2th cards in the p_1th sections in the third deal. Hence finally, the card which is in the p_2th column before the second deal, p_1th before the third deal, and p_0th, when the cards are taken up finally, will be the kth card in the pack, where

$$k = \{ (p_0 - 1)\, n^2 + (p_1 - 1)\, n + (p_2 - 1) \} + 1.$$

This justifies the rule stated above.

4. Precisely the same argument holds for a pack of n^p cards dealt into n^{p-1} rows, each of n cards, the only difference being that p deals will now be necessary. If $r_{p-1}, r_{p-2}, r_{p-3} \cdots r_0$, denote the positions which the columns pointed out by the spectator occupy before the second, third, ... pth deals, and before the final arrangement, respectively, then the position of the card thought of, is given by:

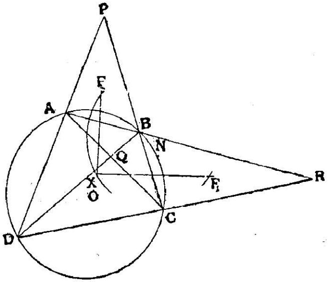
$$k = \{ (r_0 - 1)\, n^{p-1} + (r_1 - 1)\, n^{p-2} + \ldots + (r_{p-1} - 1) \} + 1.$$

As far as I am aware, this variation of the usual card trick does not appear to have found its way into books.

R. VAIDYANATHASWAMY.

On the Foci of the two parabolas through four concyclic points.

Let ABCD be a quadrilateral inscribed in a circle whose centre is O, S the circum-centre of the harmonic triangle of ABCD, and X the mean centre of A, B, C, D Through X draw XF_1, XF_2 parallel to the bisectors of the angle between the diagonals AC, BD. With N, the middle point of OS, as centre and NX as radius, describe a circle to cut the lines XF_1, XF_2 in F_1, F_2. Then F_1, F_2 are the foci of the parabolas through A, B, C, D.



The above contruction is based on the following properties of the parabola :—

(i) If a circle cut a parabola in four points ABCD, the mean centre of ABCD lies on the axis of the parabola. (This follows from the familiar relation $t_1 + t_2 + t_3 + t_4 = 0$ connecting the parameters of four concyclic points.

(ii) The chords joining the four points ABCD in pairs are equally inclined to the axis.

(iii) If a quadrangle is inscribed in a conic, its harmonic triangle is self polar w r.t. the conic.

(iv) The focus of any parabola self polar with respect to a triangle lies on its nine-pts circle and its directrix passes through the circum-centre.

(v) The mean centre of four concyclic points lies on the nine-pts. circle of the harmonic triangle. (*Vide* J. I. M. S., Question 1122).

It will now be obvious that XF_1 and XF_2 are the axes of the two parabolas through ABCD. Since O is the orthocentre and S the circum-centre of the harmonic triangle, the circle centre N and radius NX is the nine-pts circle whose intersections with the axes XF_1 and XF_2 are the foci of the parabolas.

C. N. SRINIVASIENGAR.

On the Average Value of $\frac{\phi(n)}{n}$.

Let $\phi(n)$ denote the number of integers less than and prime to n It is known that the set of numbers $\phi(n)/n$ is everywhere dense in the interval zero to one,* and that each value occurs infinitely often. In this note is shown that the average value of $\phi(n)/n$ is $6/\pi^2$ by relating this problem to another in the theory of probability.†

Let a and b be two integers chosen at random, $a \leqslant n$ but b unrestricted. Consider now the probability that a and b are prime to each other. Since a may take any one of the values $1, 2, 3, \ldots r \ldots n$, and all are equally likely the chance that a should be equal to any particular integer r is $1/n$. a being thus fixed, b may be any one of the series of natural numbers, and these when divided by $r\ (= a)$ leave remainders $0, 1, \ldots r-1$,

that is $$b \equiv 0, 1, 2, \ldots \text{ or } r-1 \pmod r$$

and all these remainders are equally likely. b will be prime to $a\ (= r)$ only when b is congruent (mod r) to any one of the $\phi(r)$ numbers $< r$ and prime to r. Hence the chance that a is prime to b is in this case $\phi(r)/r$ Remembering that a may take any value from 1 to n, and that the probability for each is $1/n$ we have

$$\frac{1}{n}\left(\frac{\phi(1)}{1} + \frac{\phi(2)}{2} + \frac{\phi(3)}{3} \cdots \frac{\phi(n)}{n}\right)$$

as the required probability and making n indefinitely large, we see that the probability ρ that when two integers are chosen at random, they should be relatively prime is

$$\underset{n \to \infty}{\text{Lt}} \ \frac{1}{n}\left(\frac{\phi(1)}{1} + \frac{\phi(2)}{3} + \frac{\phi(3)}{3} \cdots \frac{\phi(n)}{n}\right)$$

= mean asymptotic value (*valeur moyen asymptotique*) of $\frac{\phi(n)}{n}$ extended to all integers n.

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* T. Vijayaraghavan: On the set of point $\phi(n)/n$, J. I. M. S., Vol. XII, p. 98.

† I am not aware of any references to the connection between these two problems. Sylvester (*Coll. Papers*, Vol. IV, p 86) deduces the probability that two integers chosen at random be relatively prime from the asympotic behaviour of $\Sigma\,\phi(n)$. Kronecker in his *Zahlentheorie* I, p. 334 has determined the mean value of $\phi(n)/n$ but this book has not been available for reference.

But we might calculate the chance $(1 - \beta)$ that a and b may have a common factor, in quite a different manner. Let f be their highest common factor. This will be the case if

$$a \equiv 0 \pmod f \quad i.e. \quad a = a'f,$$

$$b \equiv 0 \pmod f \quad i.e. \quad b \equiv b'f,$$

and a' and b' are relatively prime.

From the preceding discussion it is easy to see that the probabilities of these three events are $\frac{1}{f}, \frac{1}{f}$, and β (since a' and b' are *any* two integers, if a and b are unrestricted). Hence the probability that a and b have a highest common factor f is $\frac{\beta}{f^2}$. Remembering that f may be any one of the series of natural numbers 2, 3, 4 ... ...

we have $$1 - \beta = \beta \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \quad \dots\right)$$

i.e. $$1 = \beta \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots\dots\right) = \beta \frac{\pi^2}{6}.$$

Hence β, the average value of $\frac{\phi(n)}{n}$ is $\frac{6}{\pi^2} = \cdot 6079$ nearly.

A. Narasinga Rao.

Solutions.

Question 977.

(M. K. KEWALRAMANI) :—In the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\dots\dots$$

if the terms are deranged in such a manner that to every 6561 positive terms, correspond 256 negative terms, the terms in each group having their original order, show that the sum of the new series is

$$\frac{\sqrt{2}}{8}\left[\pi-\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]+\log\frac{3}{2}$$

Solution by S. Audinarayanan.

If $f(n)$ be the nth term of the series and if with every $(n+\nu)$ positive terms are taken n negative terms, the sum of the first r terms $(r=2n+v)$ is

$$\sum_{1}^{2n}(-)^{n-1}f(n)+[f(2n+1)+f(2n+3)+\dots\dots$$

$$+f(2n+2v-1)]\dots\quad\text{(i)}$$

Let $nf(n)\to g$ as n becomes infinite. Then given $\varepsilon>0$ we can find an n_0 such that for $n>n_0$,

$$\frac{g-\varepsilon}{n}<f(n)<\frac{g+\varepsilon}{n},$$

If $(n+v)/n$ tends to a finite limit k as $n\to\infty$, then

$$\lim_{n\to\infty}\left[\frac{1}{2n+1}+\frac{1}{2n+3}+\dots+\frac{1}{2n+2v-1}\right]$$

$$=\lim_{n\to\infty}\int_{n}^{n+v}\frac{dx}{2x}=\tfrac{1}{2}\log k.$$

∴ The expression within brackets in (1) lies between $\frac{1}{2}(g-\varepsilon)\log k$ and $\frac{1}{2}(g+\varepsilon)\log k$.

As n becomes infinite, the deranged series will thus tend to

$$\sum_{1}^{\infty} (-)^{n-1} f(n) + \tfrac{1}{2} g \log k.$$

In the present case

$$f(n) = \frac{1}{4n-1}; \quad g = \frac{1}{4}; \quad k = \frac{6561}{256} = \left(\frac{3}{2}\right)^8$$

But $$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} \cdots = \frac{\sqrt{2}}{8}\left[\pi - \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

(*vide* Vol. X of this *Journal*, p. 442.

Hence the sum of the deranged series is

$$\frac{\sqrt{2}}{8}\left[\pi - \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right] + \log\frac{3}{2}$$

The general case is discussed by Bromwich: *Theory of Infinite Series*, p. 69.

Question 1122.

(V. Ramaswamy Iyer):—If the joins of four concyclic points A, B, C, D taken in opposite pairs intersect in P, Q, R, prove that the nine-point circle of PQR passes through the centroid of the points A, B, C, D.

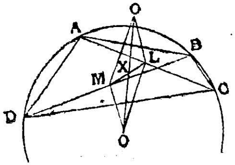
Solution and Remarks by C. N. Srinivasiengar.

The following proof seems to be simpler than those which have already appeared :—

When four points are concyclic, the eleven-point conic which is the locus of the centres of all the circum-conics of the quadrangle is a rectangular hyperbola. The centre of this eleven-point conic, (which passes through P, Q, R, and the mid-points of the sides of ABCD), is obviously the centroid of A, B, C, D. But the centres of all rectangular hyperbolas through P, Q, R lie on its nine-point circle. Hence, the theorem.

Let L and M be the mid.-points of the diagonals APC, BPD; O and X the centre of the circle ABCD and the centroid of ABCD respectively. Then, if O′ is the ortho-centre of the triangle PLM, the figure OLO′M is a parallelogram, and and since X is the mid.-point of LM, it

follows that OXO′ is a line, and OX = XO′. Now, O is the ortho-



centre of triangle PQR, and X lies on the nine-point circle. Hence O′ must lie on the circum-circle of PQR. The following corollaries follow:—

(1) O′ is the centre of the rectangular hyperbola through A, B, C, D: for the centroid of ABCD is midway between the centres of the circle and rectangular hyperbola through A, B, C, D.

(2) The pedal lines of O′ and of the point on the circle PQR diametrically opposite to O′, intersect at X.

(3) O′ is the fourth point of intersection of the circle PQR, and the eleven-point conic of ABCD.

This follows from the fact that the rectangular hyperbola through PLM, also passes through its ortho-centre O′; or otherwise, since O lies on the eleven-point conic.

(4) It is easily proved by the use of Apollonius's theorem, that the diameter d of the circum-circle of PQR is given by

$$d^2 = XP^2 + XQ^2 + XR^2 + XO^2.$$

Question 1162.

(K. J. SANJANA):—Solve the following differential equations and explain their geometrical significance.—

$$\text{(i)}\quad x^2 + y^2 - 2\frac{xy_1(1 + y_1^2)}{y_2} + \frac{2y_1(1 + y_1^2)}{y_2} = k^2.$$

$$\text{(ii)}\quad x^2 + y^2 - \frac{2xy_1(1 + y_1^2)}{y_2} + 2y\frac{(1 + y_1^2)}{y_2} + \frac{(1 + y_1^2)^3}{y_2^2} = k^2.$$

$$\text{(iii)}\quad y - mx + \frac{1 + y_1^2}{y_2} + \frac{my_1(1 + y_1^2)}{y_2} = 0,$$

where y_1 and y_2 stand for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and k and m are given constants.

Remarks by A. A. Krishnaswamy Aiyangar.

A complete analytical solution of (i) and (iii) has been given by Messrs. T. P. Trivedi and K. J. Sanjana in Vol. XV, pp. 22—24 of this *Journal.* Their solution to (ii) is however, only geometrical, and incomplete. The following is the complete analytical solution :—

Solving for y_2 we have

$$y_2 = \frac{(1+y_1^2)\{xy_1 - y \pm \sqrt{(xy_1-y)^2-(1+y_1^2)(x^2+y^2-k^2)}\}}{x^2+y^2-k^2} \quad \ldots \quad (1)$$

Put $$xy_1 - y = z(1+y_1^2)^{\frac{1}{2}}(x^2+y^2-k^2)^{\frac{1}{2}} \ldots \quad \ldots \quad (2)$$

Differentiating (2), and rearranging the terms,

$$\frac{(yy_1+x)\,y_2}{1+y_1^2} = \frac{xy_1-y}{z}\frac{dz}{dx} + \frac{(xy_1-y)(x+yy_1)}{x^2+y^2-k^2}. \quad \ldots \quad (3)$$

Eliminating y_2 from (1) and (3), using (2), we get

$$\frac{dz}{dx} = \pm \frac{\sqrt{z^2-1}\,(x+yy_1)}{x^2+y^2-k^2}. \quad \ldots \quad \ldots \quad (4)$$

If $z^2 \neq 1$, (4) gives on integration,

$$\log(z+\sqrt{z^2-1}) = \pm \tfrac{1}{2}\log(x^2+y^2-k^2) + \log C,$$

i.e. $$z = \tfrac{1}{2}\{C(x^2+y^2-k^2)^{\frac{1}{2}} + \frac{1}{C}(x^2+y^2-k^2)^{-\frac{1}{2}}\}. \quad \ldots \quad (5)$$

Substituting in (2)

$$2(xy_1 - y) = (1+y_1^2)^{\frac{1}{2}}\left\{C(x^2+y^2-k^2) + \frac{1}{C}\right\} \quad \ldots \quad (6)$$

Differentiating (6), we have

$$2xy_2 = 2C(1+y_1^2)^{\frac{1}{2}}(x+yy_1) + y_1y_2(1+y_1^2)^{-\frac{1}{2}} \times \left\{C(x^2+y^2-k) + \frac{1}{C}\right\}$$

i.e. $$2xy_2 = 2C(1+y_1^2)^{\frac{1}{2}}(x+yy_1) + 2(xy_1-y)(1+y_1^2)^{-1}y_1y_2$$

or $$y_2 = C(1+y_1^2)^{3/4},$$

removing the factor $x + yy_1$, which can be supposed not to vanish.

We, now get on integration

$$(x-H)^2+(y-G)^2=\frac{1}{C^2}. \quad \ldots \quad \ldots \quad (7)$$

The constants H, G and C cannot be all independent. To find the relation between them, differentiate (7).

We get

$$y_1=-\frac{x-H}{y-G}.$$

Substituting in (6),

$$\frac{-2(x^2+y^2-xH-yG)}{y-G}$$

$$=\frac{\{(x-H)^2+(y-G)^2\}^{\frac{1}{2}}}{y-G}\left\{C(x^2+y^2-k^2)+\frac{1}{C}\right\}$$

$\therefore\ H^2+G^2=k^2$ by (7).

The centres of the circles (7) therefore lie on $x^2+y^2=k^2$, a result deduced from geometrical considerations by Messrs. Sanjana and Trivedi.

Before integrating (4), we assumed that $z^2\neq 1$, but if $z^2=1$, we get from (2),

$$xy_1-y=\pm(1+y_1^2)^{\frac{1}{2}}(x^2+y^2-k^2)^{\frac{1}{2}} \ldots \quad \ldots (8)$$

Squaring both sides and re-arranging

$$(x+yy_1)^2=k^2(1+y_1^2)$$

$$\therefore \quad (x+yy_1)=\pm k(1+y_1^2)^{\frac{1}{2}} \ldots \quad \ldots (9)$$

$\therefore$ By (8) and (9)

$$xy_1-y=\pm\frac{x+yy_1}{k}(x^2+y^2-k^2)^{\frac{1}{2}}$$

Using the integrating factor $1/(x^2+y^2)$, we may put this in the form

$$\frac{xy_1-y}{x^2+y^2}=\pm\left\{\frac{x+yy_1}{k}(x^2+y^2-k^2)^{-\frac{1}{2}}-\frac{k(x+yy_1)(x^2+y^2+k^2)^{-\frac{1}{2}}}{x^2+y^2}\right\}$$

which gives on integration

$$\tan^{-1}\frac{y}{x}=A\pm\left(\frac{\sqrt{x^2+y^2-k^2}}{k}-\cos^{-1}\frac{k}{\sqrt{x^2+y^2}}\right),$$

which represents the involutes of the circle $x^2+y^2=k^2$ (Salmon: Higher Plane Curves, Third Edition, p. 291) as is obvious geometrically.

Question 1283.

(K. J. SANJANA, M.A.):—A conic is circumscribed to the triangle of reference:

(i) If its centre be at the point (f, g, h), prove that its equation is

$$\Sigma f(bg + ch - af) \div \alpha = 0.$$

(ii) If one focus be at (f, g, h), the equation is

$$\Sigma f(mg + nh - lf) \div \alpha = 0,$$

where l, m, n are the sides of the pedal triangle of the given focus.

Solutions by R. Gopalaswami Aiyangar and K. C. Shah.

Let (f, g, h) be the pole ot the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$ with respect to the conic

$$u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0 \quad \dots \quad \dots \quad (1)$$

Then we have

$$\lambda : \mu : \nu = vh + wg : wf + uh : ug + vf$$

whence by solving for u, v, w we have

$$u : v : w = f(g\mu + h\nu - f\lambda) : \text{two similar expressions.} \quad \dots \quad (2)$$

We deduce from (1) and (2) *that the equation of a circum-conic for which the line* $\lambda\alpha + \mu\beta + \nu\gamma = 0$ *is the polar of the point* (f, g, h) *is*

$$\Sigma f(g\mu + h\nu - f\lambda)\beta\gamma = 0 \quad \dots \quad \dots \quad (3)$$

The first part of the question follows on writing

$$\lambda : \mu : \nu = a : b : c.$$

To prove the second part we have only to show that the directrix is given by

$$l\alpha + m\beta + n\gamma = 0$$

in other words that the distances of the vertices from the directrix are proportional to $l/a : m/b : n/c$, l, m, n being the sides of the pedal triangle of the focus S. But these distances from the directrix are proportional to SA : SB : SC, while the sides l, m, n of the pedal triangle of S are given by $\frac{l}{a} : \frac{m}{b} : \frac{n}{c}$ = SA : SB : SC, whence the result follows.

Question 1312.

(B. B. BAGI):—If OA_0A_1, $A_0A_1A_2$, $A_1A_2A_3$ ... ... are similar triangles in a plane such that $A_0A_1 = k\,OA_0$ $(k < 1)$, show that if the process is continued indefinitely the point A_n tends to a definite position in the plane. Give a geometrical construction for it.

Solution by A. Narasinga Rao, M. K. Kewalramani, M. M. Mehta, P. R. Venkatakrishna Iyer, Ganesan and S. Audinarayanan.

Take O for origin and the step OA_0 as the unit length along the axis of reals on the Argand plane. The point A_1 then corresponds to a complex number $1 + z$ where $|z| = A_0A_1$ and arg. $z = \pi - \angle OA_0A_1$. Since triangles OA_0A_1 and $A_0A_1A_2$ are similar, $A_1A_2 = (A_0A_1)^2 = |z|^2$ and makes an angle $2(\pi - \angle OA_0A_1)$ with OA_0. Hence step A_1A_2 correspond to z^2 and A_2 is the point $1+z+z^2$. In the same manner it can be shown that A_n is the point $1 + z + z^2 \ldots z^n$.

Since $|z| = \dfrac{A_0A_1}{OA} = k < 1$, the series Σz^r converges to the sum $\dfrac{1}{1-z}$. The point A_n accordingly tends to a definite limit A.

To determine this point, produce A_1A_0 backwards to an equal distance A_0B. Obtain the reflexion B' of B in the line OA_0. Along OB' mark off a length $OA = 1/OB$. Then A is the required point.

If however the angles $\angle OA_0A_1$ and $\angle A_0A_1A_2$ are taken in opposite senses, it is easy to see that the points $O, A_1, A_3, A_5 \ldots$ are collinear and so also the points $A_0, A_2, A_4, A_6 \ldots$. The point A_n tends in this case to the intersection of these two lines.

Question 1348.

(S. D. CHOWLA):—Prove that

$$\frac{e^{-\pi}}{1+e^{-\pi}} + \frac{3e^{-3\pi}}{1+e^{-3\pi}} + \frac{5e^{-5\pi}}{1+e^{-5\pi}} + \ldots = \frac{1}{24}$$

$$\frac{e^{-2\pi}}{1-e^{-2\pi}} + \frac{2^5e^{-4\pi}}{1-e^{-4\pi}} + \frac{3^5e^{-6\pi}}{1-e^{-6\pi}} + \ldots = \frac{1}{504}$$

Remarks by S. Audinarayanan.

Put $m = 0$ in equation (5) page 154 of Vol. XV of this *Journal* to get the first result; and $n = 2$, $a = \pi$ in equation (2) page 153 to get the second result.

Question 1357.

(R. VAIDYANATHASWAMY):—A, B are two points, in a plane, so related that when A is fixed, the locus of B is the circle SAA′, where A′ is the point such that SA = SA′ and ∠ASA′ is a fixed angle θ, S being a fixed point. If B is fixed, what is the locus of A?

Solution by K. O. Shah, M.A.

Let any two straight lines at right-angles through S be taken as the axes of reference, and the co-ordinates of A referred to them be $(r \cos\alpha,\ r \sin\alpha)$ Then the co-ordinates of A′ are $r\cos(\alpha+\theta)$, $r\sin(\alpha+\theta)$, and the equation of the circle SAA′ is

$$\begin{vmatrix} x^2+y^2 & x & y \\ r^2 & r\cos\alpha & r\sin\alpha \\ r^2 & r\cos(\alpha+\theta) & r\sin(\alpha+\theta) \end{vmatrix} = 0$$

B satisfies this equation. If B is fixed, say at (h, k), the locus of A is found by substituting x, y, h, k for $r\cos\alpha$, $r\sin\alpha$, x, y respectively in the above equation. Hence the locus of A is

$$\begin{vmatrix} h^2+k^2 & h & k \\ x^2+y^2 & x & y \\ x^2+y^2 & x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta \end{vmatrix} = 0$$

Simplifying

$$x(k\cos\theta - h\sin\theta - k) - y(k\sin\theta + h\cos\theta - h) + \sin\theta\,(h^2+k^2) = 0$$

Therefore the locus of A is a straight line.

Question 1363.

(S. NARAYANA IYER):—In Dr. Glaisher's Notation, show that

$$(1)\quad \operatorname{cs}\left(\frac{K}{n}\right).\operatorname{cs}\left(\frac{2K}{n}\right)\ldots\ldots\operatorname{cs}\frac{(n-1)K}{n} = (1-k^2)^{\frac{n-1}{4}}.$$

$$(2)\quad \operatorname{ds}\frac{(K+iK')}{n}\operatorname{ds}\frac{2(K+iK')}{n}\ldots\ldots\operatorname{ds}\frac{(n-1)(K+iK')}{n}$$

$$= k^{n-1}\{\operatorname{cn}(K+iK')\}^{\frac{n-1}{2}},$$

Solution and Remarks by V. Thiruvenkatacharya, M.A., L.T., and Hans. R. Gupta.

We note that

$$\text{cs}\,\frac{(n-1)\,\text{K}}{n} = \text{cs}\left(\text{K}-\frac{\text{K}}{n}\right) = k'\ \text{sc}\,\frac{\text{K}}{n}.$$

$$\text{cs}\,\frac{(n-2)\,\text{K}}{n} = k'\ \text{sc}\,\frac{2\text{K}}{n}.$$

$$\ldots \qquad \ldots \qquad \ldots$$

Now the first and the last expressions cancel each other leaving k', similarly the second and the last but one; and so on.

Therefore

$$\text{cs}\,\frac{\text{K}}{n}\,\text{cs}\,\frac{2\text{K}}{n}\ \ldots\quad \ldots\ \text{cs}\,\frac{(n-1)\,\text{K}}{n} = (k')^{\frac{n-1}{2}} = (1-k^2)^{\frac{n-1}{4}}.$$

Similarly we have

$$\text{ds}\,\frac{(n-1)(\text{K}+i\text{K}')}{n} = \text{ds}\left((\text{K}+i\text{K}' - \frac{\text{K}+i\text{K}'}{n}\right)$$

$$= -ik'k\ \text{sd}\,\frac{\text{K}+i\text{K}'}{n}.$$

This multiplied by the first factor, *viz.* ds $\frac{\text{K}+i\text{K}'}{n}$, leaves $-ikk'$.

Therefore combining the second with the last but one and so on, we have

$$\text{ds}\,\frac{\text{K}+i\text{K}'}{n}\,\text{ds}\,\frac{2(\text{K}+i\text{K}'}{n}\ \ldots\ \ldots\ \text{ds}\,\frac{(n-1)(\text{K}+i\text{K}')}{n}$$

$$= \{-ikk'\}^{\frac{n-1}{2}}.$$

$$= k^{n-1}\left\{-i\frac{k'}{k}\right\}^{\frac{n-1}{2}} = k^{n-1}\,[\text{cn}\,\{\text{K}+i\text{K}'\}]^{\frac{n-1}{2}}.$$

Formulae similar to the above are given below.

(i) $$\text{dc}\,\frac{i\text{K}'}{n}\,\text{dc}\,\frac{2i\text{K}'}{n}\ \ldots\ \ldots\ \text{dc}\,\frac{(n-1)\,i\text{K}'}{n} = k^{\frac{n-1}{2}}.$$

(ii) $$\text{dn}\,\frac{\text{K}}{n}\,\text{dn}\,\frac{2\text{K}}{n}\ \ldots\ \ldots\ \text{dn}\,\frac{(n-1)\,\text{K}}{n} = (1-k^2)^{\frac{n-1}{4}}.$$

(iii) $$\text{sn}\,\frac{i\text{K}'}{n}\,\text{sn}\,\frac{2i\text{K}'}{n}\ \ldots\ \ldots\ \text{sn}\,\frac{(n-1)\,i\text{K}'}{n} = (-k)^{-\frac{n-1}{2}}.$$

(iv) $$\text{cn}\,\frac{(\text{K}+i\text{K}')}{n}\,\text{cn}\,\frac{2(\text{K}+i\text{K}')}{n}\ \ldots\ \ldots\ \text{cn}\,\frac{(n-1)(\text{K}+i'\text{K})}{n}$$

$$= \left(-i\frac{k'}{k}\right)^{\frac{n-1}{2}} = \{\text{cn}(\text{K}+i\text{K}')\}^{\frac{n-1}{2}}.$$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1444. (SELECTED):—Prove that

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\dots\dots=\frac{\theta^2}{1^3}+\frac{\theta^4}{2^3}+\frac{\theta^6}{3^3}+\dots\dots$$

where $\theta = 2\sin\frac{\pi}{10}$.

1445. (S. D. CHOWLA):—Prove Ramanujan's result:

$$\frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)}=\sum\frac{\sigma_a(n)\,\sigma_b(n)}{n^s}$$

where $\sigma_a(n)$ denotes the sum of the "ath" powers of the divisors of n,

and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

1446. (V. RAMASWAMI AIYAR):—ABC and $\alpha\beta\gamma$ are two directly similar triangles, such that the orthocentre of the former is the circumcentre of the latter. Show that the reflexions of α, β, γ in BC, CA, AB lie on a circle whose centre is the circumcentre of ABC.

1447. (V. RAMASWAMI AIYAR):—Given two circles X, X′, let ABC, A′B′C′ be two directly similar triangles inscribed in them. If AA′, BB′, CC′ cut the circle X in P, Q, R and the circle X′ in P′, Q′, R′, show that the triangle PQR is similar to the triangle P′Q′R′.

If the circles X, X′ do not intersect and L be one of their limiting points, and if the joins of A, B, C, P, Q, R to L cut the circle X again in a, b, c, p, q, r; and the joins of A′, B′, C′, P′, Q′, R′ to L cut the circle X′ again in a', b', c', p', q', r'; show that the triangle abc is similar to $p'q'r'$ and the triangle pqr to $a'b'c'$.

1448. (A. NARASINGA RAO):—Prove that the product of the lengths of the normals from a point P, (x, y) to a real algebraic curve $f(x, y) = 0$ is

$$f(x, y)\,.\,PS_1\,.\,PS_2\,.\,PS_3\,\dots\,\dots$$

to a constant *pres* and $S_1 S_2 \dots \dots$ are the several real foci of the curve.

Determine the constant when the curve is a conic section.

1449. (A. NARASINGA RAO):—Find the condition that one of the common chords of two conics $a_x^2 = 0$ and $b_x^3 = 0$ may be a tangent to a third conic $c_x^2 = 0$.

Point out how this result may be utilised to find the locus of the foci of parabolic sections of a given conicoid.

1450. (N. DURAIRAJAN):—Show that if three consecutive tangents be drawn to a curve, the circum-radius of the triangle so formed is ultimately $\frac{1}{4}$ of the radius of curvature of the curve at the point, and that the circum-centre and ortho-centre of the triangle divide harmonically the line joining the point and its centre of curvature.

1451. (N. DURAIRAJAN):—If G be the centre and S a focus of the maximum inscribed ellipse of a triangle and if GS be produced to S′ so that GS′ = √3 GS, then prove that

$$S'A . S'B . S'C = GA . GB . GC.$$

1452. (SELECTED):—ABCDEF is a hexagon inscribed in a conic S so that any two consecutive vertices are conjugate points with respect to another conic S′, prove that

$$\theta^3 - 4\,\Delta\,(\theta\theta' - 2\,\Delta\Delta') = 0.$$

1453. (K. J. SANJANA, M.A.):—(1) It is easily seen by twice differentiating the equation

$$k\left(\frac{d^2y}{dx^2}\right)^{\frac{8}{3}} + l\left(\frac{d^2y}{dx^2}\right)^{\frac{10}{3}} = \left(\frac{d^3y}{dx^3}\right)^2 \qquad \ldots \quad (A)$$

and eliminating the constants linearly, that we get the general differential equation of the fifth order of any conic section. Determine whether by starting from this fifth order differential equation and integrating it twice, we can directly obtain the equation (A) in which k and l are arbitrary constants

1454. (K. J. SANJANA, M.A.):—It is not difficult to prove that the equation

$$\frac{dy}{dx} = \frac{m\,(a^2 - x^2) - xy}{a^2 - x^2 + mxy} \qquad \ldots \quad (B)$$

represents the curves which cut at the angle $\tan^{-1} m$ all central conics having a common transverse axis. Determine in the case when $m = 1$, whether the above equation on direct integration can be made to represent this group of central conics.

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LIST OF JOURNALS RECEIVED IN THE LIBRARY

during September and October 1926.

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1 American Journal of Mathematics, **48**, 3 (2 copies).

2 American Mathematical Monthly, **33**, 5 & 6. (2 copies).

3 Annales de L'Ecole Normale Supérieure, **6**, 9.

4 Astrophysical Journal, **63**, 5 and **64**, 1.

5 Bulletin of the American Mathematical Society, **32**, 4.

6 Crelle's Journal, **156**, 1.

7 Mathematical Gazette, **13**, 184, (2 copies).

8 Messenger of Mathematics, **56**, 1, 2, 3, 4 (2 copies each).

9 Monthly Notices of the Royal Astronomical Society, **86**, 7 & 8.

10 Philosophical Magazine, **2**, 9 & 10.

11 Philosophical Transactions of the Royal Society, **226**, 636, 637, 638, 639.

12 Popular Astronomy, **34**, 7 (2 copies).

13 Proceedings of the Royal Society of London, **111**, 757, **112**, 760, 761, 762.

14 Transactions of the American Mathematical Society, **28**, 2.

15 Transactions of the Cambridge Philosophical Society, **23**, 9.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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